

ON q -NORMAL OPERATORS AND QUANTUM COMPLEX PLANE.

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ABSTRACT. For $q > 0$ let \mathcal{A} denote the unital $*$ -algebra with generator x and defining relation $xx^* = qxx^*$. Based on this algebra we study q -normal operators, the complex q -moment problem, positive elements and sums of squares.

1. INTRODUCTION

Suppose that q is a positive real number. A densely defined closed linear operator X on a Hilbert space is called q -normal if

$$(1) \quad XX^* = qX^*X.$$

This and other classes of q -deformed operators have been introduced and investigated by S. Ota [Ota], see e.g. [OS2]. In this paper we continue the study of q -normal operators. Further, let \mathcal{A} denote the unital complex $*$ -algebra with single generator x and defining relation

$$(2) \quad xx^* = qx^*x.$$

The algebra \mathcal{A} appears in the theory of quantum groups where it is considered as the coordinate algebra of the q -deformed complex plane, or briefly, of the *complex q -plane*.

Let us set for a moment $q = 1$. Then the q -normal operators are precisely the normal operators and \mathcal{A} is a complex polynomial algebra $\mathbb{C}[x, \bar{x}]$. It is well known that there is a close relationship between various important algebraic and analytic problems:

- the complex moment problem,
- the extension of formally normal operators to normal operators (possibly in a larger Hilbert space),
- the characterization of well-behaved representations of the $*$ -algebra $\mathbb{C}[x, \bar{x}]$,
- the representation of positive polynomials as sums of squares (motivated by 17-th Hilbert problem)

The aim of the present paper is to begin a study of these problems and their interplay in the q -deformed case.

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We now discuss the contents of this paper. Section 2 deals with q -normal operators. After giving some equivalent characterizations of q -normality we prove a structure theorem for q -normal operators (Theorem 1) which is the counter-part of the spectral theorem for unbounded normal operators.

Section 3 deals with positive q -polynomials and their possible representations as sums of squares. We define the cone \mathcal{A}_+ of *positive* elements to be the set of elements which are mapped into positive symmetric operators by all well-behaved $*$ -representations of the $*$ -algebra \mathcal{A} . A $*$ -representation π of \mathcal{A} with domain $\mathcal{D}(\pi)$ is called *well-behaved* if there exists a q -normal operator X such that $\mathcal{D}(\pi) = \cap_{n=1}^{\infty} \mathcal{D}(X^n)$ and $\pi(x) = X[\mathcal{D}(\pi)]$. Theorem 2 states that for each positive $q \neq 1$ there exists a polynomial $p_q \in \mathbb{R}[t]$ of degree four such that the element $f := p_q(x + x^*)$ is in \mathcal{A}_+ , but f is not a sum of squares in \mathcal{A} . In contrast we prove that if $p \in \mathbb{R}[t]$ and $p(x^*x) \in \mathcal{A}_+$, then $p(x^*x)$ is always a sum of squares.

In Section 4 we study a generalization of the complex moment problem to the $*$ -algebra \mathcal{A} . Let F be a linear functional on \mathcal{A} . We say that F is a *q -moment functional* if there exists a well-behaved $*$ -representation π of \mathcal{A} and a vector $\varphi \in \mathcal{D}(\pi)$ such that $F(a) = \langle \pi(a)\varphi, \varphi \rangle$ for $a \in \mathcal{A}$. (In Section 4 we use Theorem 1 to give a formulation of q -moment functionals in terms of measures.) Further, F is called *positive* if $F(f^*f) \geq 0$ for all $f \in \mathcal{A}$ and *strongly positive* if $F(f) \geq 0$ for all $f \in \mathcal{A}_+$. Then, by Theorem 3, a linear functional on \mathcal{A} is a q -moment functional if and only if it is strongly positive. This result can be considered as the counter-part of Haviland's theorem. In contrast, by Theorems 4 and 5, there exists a positive linear functional F on \mathcal{A} which is not a q -moment functional and a formally q -normal operator which has no q -normal extensions.

In the commutative case $q = 1$ the solution of Hilbert's 17-th problem (see e.g. [M]) implies that positive polynomials of $\mathbb{C}[x, \bar{x}]$ are sums of squares of rational functions. In Section 5 we prove a Positivstellensatz (Theorem 6) which states, roughly speaking, that *strictly positive* elements of \mathcal{A}_+ can be represented as sums of squares by allowing "nice" denominators.

We close this introduction by collecting some definitions and notations.

Operators in Hilbert space. For a linear operator A on a Hilbert space operator we denote by $\mathcal{D}(A)$, $\text{Ran} A$, \overline{A} and A^* denote its *domain*, its *range*, its *closure* and its *adjoint*, respectively, and we set $\mathcal{D}^\infty(A) := \cap_{n=1}^{\infty} \mathcal{D}(A^n)$. A *core* of a closed operator A is a linear subset $\mathcal{D}_0 \subseteq \mathcal{D}(A)$ such that the closure of $A \upharpoonright \mathcal{D}_0$ coincides with A . If A is self-adjoint, then $\mathcal{D}^\infty(A)$ is a core of A .

A number $\lambda \in \mathbb{C}$ is called a *regular point* for an operator A if there exists $c_\lambda > 0$ such that $\|(A - \lambda I)\varphi\| \geq c_\lambda \|\varphi\|$ for all $\varphi \in \mathcal{D}(A)$.

If A_i , $i \in I$, are linear operators on a Hilbert space \mathcal{H}_i , the *direct sum* $\oplus_{i \in I} A_i$ denotes the operator on $\mathcal{H} := \oplus_{i \in I} \mathcal{H}_i$ defined by $(\oplus_{i \in I} A_i)(\varphi_i)_{i \in I} := (A_i \varphi_i)_{i \in I}$ for $(\varphi_i)_{i \in I}$ in

$$\mathcal{D}(\oplus_{i \in I} A_i) := \{(\varphi_i)_{i \in I} : \varphi_i \in \mathcal{D}(A_i), (\varphi_i)_{i \in I} \in \mathcal{H} \text{ and } (A_i \varphi_i)_{i \in I} \in \mathcal{H}\}.$$

$*$ -Algebras and $*$ -representations. By a *$*$ -algebra* we mean a complex associative algebra \mathcal{A} equipped with a mapping $a \mapsto a^*$ of \mathcal{A} into itself, called the *involution* of \mathcal{A} , such that

$(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. In this paper each $*$ -algebra \mathcal{A} has an identity element denoted by $\mathbf{1}_{\mathcal{A}}$ or $\mathbf{1}$.

An element of the form $\sum_{j=1}^n a_j^* a_j$, where $a_1, \dots, a_n \in \mathcal{A}$ is called a *sum of squares* in \mathcal{A} . The set of all sums of squares is denoted by $\sum \mathcal{A}^2$.

We use some terminology and results from unbounded representation theory in Hilbert space (see e.g. in [S4]). Let \mathcal{D} be a dense linear subspace of a Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$. A $*$ -representation of a $*$ -algebra \mathcal{A} on \mathcal{D} is an algebra homomorphism π of \mathcal{A} into the algebra $\mathcal{L}(\mathcal{D})$ of linear operators on \mathcal{D} such that $\pi(\mathbf{1}) = I_{\mathcal{D}}$ and $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the *domain* of π and write $\mathcal{H}(\pi) := \mathcal{H}$. A $*$ -representation is *faithful* if $\pi(a) = 0$ implies $a = 0$.

We say that an element $a = a^* \in \mathcal{A}$ is *positive* in a $*$ -representation π if $\langle \pi(a)\varphi, \varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}(\pi)$.

Suppose that π is a $*$ -representation of \mathcal{A} . The *graph topology* of π is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$. Then π is *closed* if and only if $\mathcal{D}(\pi) = \bigcap_{a \in \mathcal{A}} \overline{\mathcal{D}(\pi(a))}$. We say that π is *strongly cyclic* if there exists a vector $\xi \in \mathcal{D}(\pi)$ such that $\pi(\mathcal{A})\xi$ is dense in $\mathcal{D}(\pi)$ in the graph topology of π .

A linear functional $F : \mathcal{A} \rightarrow \mathbb{C}$ on a $*$ -algebra \mathcal{A} is *positive* if $F(\sum \mathcal{A}^2) \geq 0$. Every positive functional F has a GNS representation (see e.g. [S4]), that is, there exists a $*$ -representation π_F and with cyclic vector φ such that $F(a) = \langle \pi_F(a)\varphi, \varphi \rangle$ for $a \in \mathcal{A}$.

2. q -NORMAL OPERATORS

In what follows q is a positive real number. Recall the definition of a q -normal operator, see e.g. [Ota].

Definition 1. A densely defined operator X on a Hilbert space \mathcal{H} is a q -normal operator if $\mathcal{D}(X) = \mathcal{D}(X^*)$ and $\|X^*f\| = \sqrt{q}\|Xf\|$, $f \in \mathcal{D}(X)$.

A q -normal operator with $q = 1$ is normal. Each q -normal operator X is closed. Indeed, since $\|X^*f\| = \sqrt{q}\|Xf\|$, the graph norms of X and X^* are equivalent. Therefore, since X^* is closed, X is also closed. It also implies that $\ker X = \ker X^*$.

The following proposition collects different characterizing properties of q -normal operators, cf. Chapter 2 in [OS].

Proposition 1. Let X be a closed operator on a Hilbert space \mathcal{H} and let $X = UC$ be its polar decomposition. The following statements are equivalent:

- (i) X is q -normal,
- (ii) $XX^* = qX^*X$,
- (iii) $UC^2U^* = qC^2$,
- (iv) $UCU^* = q^{1/2}C$,
- (v) $UE_C(\Delta)U^* = E_C(q^{-1/2}\Delta)$ for each Borel $\Delta \subseteq \mathbb{R}_+$,
- (vi) $Uf(C)U^* = f(q^{1/2}C)$ for every Borel function f on \mathbb{C} .

Proof. (i) \Rightarrow (ii) : We use some basic properties of quadratic forms associated with positive operators, see e.g. [RS]. Introduce the quadratic forms $\mathbf{t}_1[\varphi, \varphi] := \langle X^*\varphi, X^*\varphi \rangle$ and $\mathbf{t}_2[\varphi, \varphi] := q\langle X\varphi, X\varphi \rangle$, $\mathcal{D}[\mathbf{t}_1] = \mathcal{D}(X^*)$, $\mathcal{D}[\mathbf{t}_2] = \mathcal{D}(X)$. Let X be q -normal. Then X is closed and

$$\mathbf{t}_1[\varphi, \varphi] = \|X^*\varphi\|^2 = q\|X\varphi\|^2 = \mathbf{t}_2[\varphi, \varphi].$$

Thus, we get $\mathbf{t}_1 = \mathbf{t}_2$. Since X and X^* are closed, \mathbf{t}_1 and \mathbf{t}_2 are closed. The operators associated with \mathbf{t}_1 and \mathbf{t}_2 are XX^* and qX^*X respectively. Hence $XX^* = qX^*X$.

(ii) \Rightarrow (iii) : Since $X = UC$ is a polar decomposition, we have $X^* = CU^*$ and $\ker U = \ker C$. Then equation (ii) implies $UC^2U^* = qCU^*UC = qC^2$.

(iii) \Rightarrow (iv) : Equation (iii) implies that for $\varphi \in \mathcal{D}(C^2)$ holds

$$q\|C\varphi\|^2 = q\langle C^2\varphi, \varphi \rangle = \langle UC^2U^*\varphi, \varphi \rangle = \langle CU^*\varphi, CU^*\varphi \rangle = \|CU^*\varphi\|^2.$$

It implies $\ker U^* \subseteq \ker C$. On the other hand, if $\varphi \in \ker C$, then by the last equation $U^*\varphi \in \ker C = \ker U$. That is $UU^*\varphi = 0$, which implies $U^*\varphi = 0$. Hence $\ker U^* = \ker C = \ker U$. Restricting U, U^*, C onto $(\ker C)^\perp$ we can assume that U is unitary. Then relation (iii) defines a unitary equivalence of C^2 and qC^2 . Hence, the square roots C and $q^{1/2}C$ are unitary equivalent and we get (iv).

(iv) \Rightarrow (v) : As in the previous case we can assume that U is unitary. Then C and $q^{1/2}C$ are unitary equivalent and for every Borel $\Delta \subseteq \mathbb{R}$ we get

$$UE_C(\Delta)U^* = E_{q^{1/2}C}(\Delta) = E_C(q^{-1/2}\Delta).$$

(v) \Rightarrow (i) : Note that (v) implies that U and U^* commute with $E_C(\{0\})$, that is $\ker C$ is invariant under U and U^* . Considering the restriction of X (resp. U and C) onto $(\ker C)^\perp$ we can assume that U is unitary. Then (v) means that C and $q^{1/2}C$ are unitarily equivalent, namely $UCU^* = q^{1/2}C$, which implies $CU^* = q^{1/2}U^*C$. Using the latter we get $\mathcal{D}(X) = \mathcal{D}(UC) = \mathcal{D}(U^*C) = \mathcal{D}(CU^*) = \mathcal{D}(X^*)$ and

$$\langle CU^*\varphi, CU^*\varphi \rangle = q\langle U^*C\varphi, U^*C\varphi \rangle = q\langle UC\varphi, UC\varphi \rangle, \quad \varphi \in \mathcal{D}(X)$$

which implies $\|X^*\varphi\| = q^{1/2}\|X\varphi\|$.

(v) \Rightarrow (vi) : Follows from the computation

$$Uf(C)U^* = U\left(\int f(\lambda)dE_C(\lambda)\right)U^* = \int f(\lambda)dE_C(q^{-1/2}\lambda) = \int f(\lambda)dE_{q^{1/2}C}(\lambda) = f(q^{1/2}C).$$

(vi) \Rightarrow (v) : Follows by setting $f(\lambda) = \mathbf{1}_\Delta(\lambda)$. □

We provide a basic example of a q -normal operator for $q \neq 1$. Put

$$(3) \quad \Delta_q = \begin{cases} [1, q^{1/2}), & \text{if } q > 1, \\ (q^{1/2}, 1], & \text{if } q < 1. \end{cases}$$

Let μ be a Borel measure on $\mathbb{R}_+ = [0, +\infty)$ such that

$$(4) \quad \mu(\Delta) = \mu(q^{1/2}\Delta) \text{ for all Borel subsets } \Delta \subseteq \mathbb{R}_+.$$

Since $(0, +\infty) = \cup_{k \in \mathbb{Z}} q^k \Delta_q$, the measure μ is uniquely defined by its restriction onto the subspace $\Delta_q \cup \{0\} \subset \mathbb{R}_+$. Define an operator X_μ , on the Hilbert space $\mathcal{H}_\mu := L_2(\mathbb{R}_+, d\mu)$ as follows.

$$(5) \quad (X_\mu \varphi)(t) := q^{1/2} t \varphi(q^{1/2} t), \quad \mathcal{D}(X_\mu) = \{\varphi(t) \in \mathcal{H}_\mu \mid t \varphi(t) \in \mathcal{H}_\mu\}.$$

Let $\mathcal{H}_{\mu,0} = \{f \in \mathcal{H}_\mu \mid f \equiv 0, \text{ a.e. on } (0, +\infty)\}$. Then $\mu(\{0\}) = 0$ if and only if $\mathcal{H}_{\mu,0} = \{0\}$. We prove the following

Proposition 2.

- (i) The operator X_μ in (5) is a well-defined q -normal operator with $\ker X_\mu = \mathcal{H}_{\mu,0}$.
- (ii) The adjoint operator X_μ^* is defined by

$$(X_\mu^* \varphi)(t) = t \varphi(q^{-1/2} t), \quad \mathcal{D}(X_\mu^*) = \mathcal{D}(X_\mu).$$

- (iii) Let $X_\mu = U_\mu C_\mu$ be the polar decomposition of X_μ . Then

$$(6) \quad (C_\mu \psi)(t) := t \psi(t), \quad (U_\mu \varphi)(t) := \begin{cases} \varphi(q^{1/2} t), & \text{for } t \neq 0, \\ 0, & \text{for } t = 0. \end{cases}$$

where $\varphi \in \mathcal{H}, \psi \in \mathcal{D}(C_\mu) = \mathcal{D}(X_\mu)$.

Proof. It follows from (4) that U_μ is a well-defined partial isometry. Further, we have $X_\mu = U_\mu C_\mu$ and $\ker C_\mu = \ker U_\mu = \mathcal{H}_{\mu,0}$. Since C_μ is a positive self-adjoint operator, $X_\mu = U_\mu C_\mu$ is a polar decomposition of X_μ . Since U_μ is bounded, we have $X_\mu^* = C_\mu U_\mu^*$. Together with (4) it implies $\mathcal{D}(X_\mu^*) = \mathcal{D}(C_\mu) = \mathcal{D}(X_\mu)$. For $\varphi \in \mathcal{D}(X_\mu)$ we calculate using (4):

$$\|X_\mu \varphi\| = \|t \varphi(t)\| \quad \text{and} \quad \|X_\mu^* \varphi\| = q^{1/2} \|t \varphi(t)\|.$$

Thus, X_μ is q -normal. □

The following theorem can be viewed as an analogue of the spectral theorem for normal operators (see e.g. Theorem VII.3 in [RS]).

Theorem 1. Let X be a q -normal operator on a Hilbert space \mathcal{H} . Then there exists a family of Borel measures μ_i , $i \in I$ on \mathbb{R}_+ satisfying $\mu_i(x) = \mu_i(q^{1/2}x)$ such that X is unitarily equivalent to the direct sum of operators X_{μ_i} defined by (5).

Proof. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ denote $\ker X = \ker X^*$. The restriction $X \upharpoonright \mathcal{H}_0$ is a direct sum of copies of X_{μ_0} where $\mu_0(\mathbb{R}_+) = \mu(\{0\}) = 1$. The restriction $X_1 = X \upharpoonright \mathcal{H}_0^\perp$ is again a q -normal operator with $X_1^* = X^* \upharpoonright \mathcal{H}_0^\perp$. Without loss of generality, we can assume that $\ker X = \ker X^* = \{0\}$. Let $X = UC$ be the polar decomposition of X . Since $\ker X = \{0\}$, U is unitary.

Let $\mathcal{K} = \text{Ran } E_C(\Delta_q)$. Then \mathcal{K} is invariant under C and we denote by D the restriction $C \upharpoonright \mathcal{K}$. Further, let $\mathcal{K} = \oplus_{i \in I} \mathcal{K}_i$ an arbitrary orthogonal sum decomposition such that \mathcal{K}_i is invariant under D and let $D_i = D \upharpoonright \mathcal{K}_i$, $i \in I$. We show that there is a corresponding orthogonal sum decomposition of $X = \oplus_{i \in I} X_i$.

It follows from Proposition 1,(v) that

$$(7) \quad U^k (\text{Ran} E_C(\Delta_q)) = \text{Ran} E_C(q^{-k/2} \Delta_q), \quad k \in \mathbb{Z}.$$

Since $(0, +\infty)$ is a disjoint union of $q^{k/2} \Delta_q$, $k \in \mathbb{Z}$, we get the following direct sum decomposition

$$\begin{aligned} \mathcal{H} &= \text{Ran} E_C((0, +\infty)) = \bigoplus_{k \in \mathbb{Z}} \text{Ran} E_C(q^{k/2} \Delta_q) = \\ &= \bigoplus_{k \in \mathbb{Z}} U^{*k} \text{Ran} E_C(\Delta_q) = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{i \in I} U^{*k} \mathcal{K}_i = \bigoplus_{i \in I} \bigoplus_{k \in \mathbb{Z}} \mathcal{K}_{i,k} = \bigoplus_{i \in I} \mathcal{H}_i, \end{aligned}$$

where $\mathcal{K}_{i,k} = U^{*k} \mathcal{K}_i$ and $\mathcal{H}_i = \bigoplus_{k \in \mathbb{Z}} \mathcal{K}_{i,k}$. For $i \in I, k \in \mathbb{Z}$ we define the operators

$$(8) \quad D_{i,k} = U^{*k} (q^{k/2} D_i) U^k : \mathcal{K}_{i,k} \rightarrow \mathcal{K}_{i,k}.$$

Then for each $\varphi \in \mathcal{K}_{i,k}$ holds $U^k \varphi \in \mathcal{K}_i$ and we calculate using Proposition 1, (iv)

$$D_{i,k} \varphi = q^{k/2} U^{*k} D_i (U^k \varphi) = q^{k/2} U^{*k} C (U^k \varphi) = C U^{*k} (U^k \varphi) = C \varphi.$$

It implies that $\mathcal{K}_{i,k}$ is invariant under C and $C \upharpoonright \mathcal{K}_{i,k} = D_{i,k}$ for all $i \in I, k \in \mathbb{Z}$. Put $C_i = \bigoplus_{k \in \mathbb{Z}} D_{i,k}$, $i \in I$. Then C_i , $i \in I$ are self-adjoint with $\text{Ran} C_i \subseteq \mathcal{H}_i$ and $\bigoplus_{i \in I} C_i = C$ in particular, $C_i = C \upharpoonright \mathcal{H}_i$. The subspaces \mathcal{H}_i , $i \in I$ are invariant under U by definition of \mathcal{H}_i . We denote by U_i the restriction of U onto \mathcal{H}_i , so that $U = \bigoplus_{i \in I} U_i$. Thus we get $X = UC = \bigoplus_{i \in I} U_i C_i$.

Using Zorn's Lemma we can choose D_i to be cyclic with cyclic vectors $\psi_i \in \mathcal{K}_i$. Since $\sigma(D_i) \subseteq \Delta_q$, there exist unitary operators $V_i : \mathcal{K}_i \rightarrow L_2(\Delta_q, d\mu_i)$ such that

$$(9) \quad (V_i D_i V_i^* f)(t) = t f(t), \quad f(t) \in L_2(\Delta_q, d\mu_i),$$

where $\mu_i(\cdot) = \langle E_{D_i}(\cdot) \psi_i, \psi_i \rangle$, see e.g. Chapter VII in [RS].

It follows from (8) that operators $D_{i,k}$ are cyclic on $\mathcal{K}_{i,k}$, with cyclic vectors $\psi_{i,k} := U^{*k} \psi_i$, $i \in I, k \in \mathbb{Z}$. By (8) we also have $\sigma(D_{i,k}) \subseteq q^{k/2} \Delta_q$. We calculate the corresponding measures $\mu_{i,k}(\cdot) = \langle E_{D_{i,k}}(\cdot) \psi_{i,k}, \psi_{i,k} \rangle$ using the unitary equivalence (8):

$$\begin{aligned} \mu_{i,k}(q^{k/2} \Delta) &= \langle E_{D_{i,k}}(q^{k/2} \Delta) U^{*k} \psi_i, U^{*k} \psi_i \rangle = \langle U^k E_{D_i}(q^{k/2} \Delta) U^{*k} \psi_i, \psi_i \rangle = \\ (10) \quad &= \langle E_{q^{k/2} D_i}(q^{k/2} \Delta) \psi_i, \psi_i \rangle = \langle E_{D_i}(\Delta) \psi_i, \psi_i \rangle = \mu_i(\Delta), \end{aligned}$$

for each Borel set $\Delta \subseteq \Delta_q$. For every $k \in \mathbb{Z}$ we define an operator:

$$W_k : L_2(\Delta_q, d\mu_i) \rightarrow L_2(q^{k/2} \Delta_q, d\mu_{i,k}), \quad (Wf)(t) = f(q^{-k/2} t).$$

It follows from (10) that W_k are unitary. Further, for each $i \in I, k \in \mathbb{Z}$ we define unitary operators $V_{i,k} = W_k V_i U^k$, $V_{i,k} : \mathcal{K}_{i,k} \rightarrow L_2(q^{k/2} \Delta_q, d\mu_{i,k})$. Equations (8),(9) imply that

$$(V_{i,k} D_{i,k} V_{i,k}^* f)(t) = t f(t) \text{ for } f \in L_2(q^{k/2} \Delta_q, d\mu_{i,k}).$$

Since $\text{supp} \mu_{i,k} \subseteq q^{k/2} \Delta_q$ are disjoint for different $k \in \mathbb{Z}$, we can define a Borel measure $\tilde{\mu}_i := \sum_{k \in \mathbb{Z}} \mu_{i,k}$ on \mathbb{R}_+ which satisfies (4). Then $\tilde{V}_i = \bigoplus_{k \in \mathbb{Z}} V_{i,k}$ is a unitary operator from \mathcal{H}_i to $L_2(\mathbb{R}_+, \tilde{\mu}_i)$ such that $\tilde{V}_i C_i \tilde{V}_i^* = C_{\tilde{\mu}_i}$, where $C_{\tilde{\mu}_i}$, $i \in I$ are defined by (6). Using (8) and (9)

we get $\tilde{V}_i U_i \tilde{V}_i^* = U_{\tilde{\mu}_i}$, where $U_{\tilde{\mu}_i}$, $i \in I$ are defined by (6). Hence, $X = \oplus_{i \in I} X_i$, where every $X_i = U_i C_i$ is unitary equivalent to $X_{\tilde{\mu}_i}$, $i \in I$. \square

Definition 2. We say that a q -normal operator X is *reducible* if $\mathcal{D}(X) = \mathcal{D}_1 \oplus \mathcal{D}_2$, $\mathcal{D}_i \neq 0$ and $\mathcal{D}_1, \mathcal{D}_2$ are invariant under X . Otherwise we say that X is *irreducible*.

For irreducible q -normal operators we obtain the following description, see also [OS], p.71.

Proposition 3. *Let X be a non-zero irreducible q -normal operator on a Hilbert space \mathcal{H} . Then there exists a unique $\lambda \in \Delta_q$, and unique orthonormal base $\{e_k\}_{k \in \mathbb{Z}}$ in \mathcal{H} such that*

$$(11) \quad X e_k = \lambda q^{-k/2} e_{k+1}, \quad X^* e_k = \lambda q^{-(k-1)/2} e_{k-1}, \quad k \in \mathbb{Z}.$$

Proof. Since X , $X \neq 0$ is irreducible, by Theorem 1 we get $X = X_\mu$ for some Borel measure μ on $(0, +\infty)$ satisfying (4). It follows from the proof of Theorem 1 that operator $D = C_\mu E_{C_\mu}(\Delta_q)$ is irreducible, i.e. one-dimensional. In particular $\text{supp} \mu \cap \Delta_q$ consists of a singular point $\lambda \in \Delta_q$. Put

$$e_k = \mathbb{1}_{\{\lambda q^{-k/2}\}} (\mu(\{\lambda q^{-k/2}\}))^{-1}, \quad k \in \mathbb{Z}.$$

Direct computations show that (11) is satisfied. Further, one can check that every bounded operator which commutes with X and X^* is a multiple of identity. Hence X is irreducible. \square

Below we will often use the following

Lemma 1. *Let $q > 0$, X be a q -normal operator on a Hilbert space \mathcal{H} and let $X = UC$ be its polar decomposition.*

(i) *For all $m, n \in \mathbb{N}_0$*

$$(12) \quad X^{*m} X^n = q^{(m^2+m-n^2+n-2mn)/4} U^{n-m} C^{m+n},$$

*Where U^{-k} denotes U^{*k} , $k \in \mathbb{N}$. In particular, $\mathcal{D}^\infty(X) = \cap_{m,n \in \mathbb{N}} \mathcal{D}(X^{*m} X^n) = \mathcal{D}^\infty(C)$ is dense in \mathcal{H} .*

(ii) *The set $\mathcal{D}^\infty(X)$ is a core of $X^{*m} X^n$ for all $m, n \in \mathbb{N}_0$.*

(iii) *The set $\mathcal{D}^\infty(X)$ is invariant under U and U^* .*

Proof. (i) : By Proposition 1, (iv) we have $UC = q^{1/2} CU$, which implies

$$\begin{aligned} X^{*m} X^n &= (CU^*)^m (UC)^n = q^{[(m(m+1))/4 - (n(n-1))/4]} U^{*m} C^m U^n C^n = \\ &= q^{(m^2+m-n^2+n-2mn)/4} U^{n-m} C^{m+n}. \end{aligned}$$

(ii) : Since $\mathcal{D}^\infty(X) = \mathcal{D}^\infty(C)$ and C is self-adjoint, $\mathcal{D}^\infty(X)$ is a core of C^m , $m \in \mathbb{N}$. It follows from (12) that $\|X^{*m} X^n \varphi\| = q^{(m^2+m-n^2+n-2mn)/4} \|C^{m+n} \varphi\|$, $\varphi \in \mathcal{D}^\infty(C)$, which implies the assertion.

(iii) : By Proposition 1, (iv), we have $UC^k = q^{k/2} C^k U$, $k \in \mathbb{Z}$. Hence $\mathcal{D}(C^k)$ is invariant for U for all $k \in \mathbb{Z}$, i.e. \mathbb{C}^∞ is invariant for U . In the same way one shows that $\mathcal{D}^\infty(X)$ is invariant for U^* . \square

3. POSITIVE q -POLYNOMIALS

Recall that

$$\mathcal{A} = \mathbb{C}\langle x, x^* \mid xx^* = qx^*x \rangle,$$

where q is a positive real number. Since the set $\{x^{*m}x^n; m, n \in \mathbb{N}_0\}$ is a vector space basis of \mathcal{A} , each element $f \in \mathcal{A}$ can be written uniquely as $f = \sum_{m,n} \alpha_{mn} x^{*m}x^n$, where $\alpha_{mn} \in \mathbb{C}$. We define the *degree* of f by $\deg f := \max\{m+n \mid \alpha_{mn} \neq 0\}$. We will also refer to an element f of \mathcal{A} as a q -polynomial and write $f = f(x, x^*)$. If X is a q -normal operator, then $f(X, X^*)$ denotes the operator $\sum_{m,n} \alpha_{mn} X^{*m}X^n$.

Definition 3. An element $f = f^* \in \mathcal{A}$ is called *positive* if

$$\langle f(X, X^*)\varphi, \varphi \rangle \geq 0 \text{ for } \varphi \in \mathcal{D}^\infty(X)$$

for every q -normal operator X and every $\varphi \in \mathcal{D}^\infty(X)$. The set of positive elements of \mathcal{A} is denoted by \mathcal{A}_+ .

With this notion of positivity one can develop a non-commutative real algebraic geometry on the complex q -plane. In this section we investigate positive elements and sum of squares in \mathcal{A} . In Section 5 we prove a strict Positivstellensatz for \mathcal{A} .

Definition 4. A $*$ -representation π of \mathcal{A} is called *well-behaved* if there is a q -normal operator X such that $\mathcal{D}^\infty(X) = \mathcal{D}(\pi)$ and $\pi(x) = X \upharpoonright \mathcal{D}^\infty(X)$.

By Lemma 1 the domain $\mathcal{D}^\infty(X)$ is a core of a q -normal operator X , so there is a one-to-one correspondence between q -normal operators and well-behaved representations of \mathcal{A} .

Further, an element $f = f^* \in \mathcal{A}$ is in \mathcal{A}_+ if and only if $\pi(f) \geq 0$ for every well-behaved $*$ -representation π . Thus our definition of positive elements fits into the definition of positivity via $*$ -representations proposed in [S2].

Remarks. 1. If μ is a positive Borel measure on \mathbb{R}_+ satisfying (4) and X_μ is the q -normal operators defined by (5), we denote the corresponding well-behaved $*$ -representation of \mathcal{A} by π_μ . That is,

$$(13) \quad \pi_\mu(x) = X_\mu \upharpoonright \mathcal{D}^\infty(X_\mu), \quad \mathcal{D}^\infty(X_\mu) = \mathcal{D}(\pi_\mu).$$

2. The $*$ -algebra \mathcal{A} has a natural \mathbb{Z} -grading given by $\deg x = 1$ and $\deg x^* = -1$. In [SS] a notion of well-behaved $*$ -representations was introduced for class of group graded $*$ -algebras which contains \mathcal{A} . It can be shown that Definition 4 is equivalent to corresponding definition of well-behavedness in [SS], see Definition 11 therein.

Suppose that $p \in \mathbb{R}[t]$. We consider the following questions:

$$\text{When } p(x + x^*) \in \mathcal{A}_+? \quad \text{When } p(x + x^*) \in \sum \mathcal{A}^2?$$

First we consider the case when $\deg p = 2$.

Below we will need the following

Lemma 2. Let $f = \sum_{m,n} \alpha_{mn} X^{*m} X^n \in \mathcal{A}$, $\deg f = 2N$, $N \in \mathbb{N}$. Denote by w_N the column vector of monomials

$$w_N = (1 \quad x \quad x^* \quad x^2 \quad x^*x \quad x^{*2} \quad \dots \quad x^{*N})^T,$$

and let $w_N^* = (1 \quad x \quad x^* \quad x^{*2} \quad x^*x \quad x^2 \dots)$. Then f is a sum of squares in \mathcal{A} if and only if there exists a positive semidefinite $(N+1)(N+2)/2 \times (N+1)(N+2)/2$ complex matrix C such that $f = w_N^* C w_N$.

Proof. Assume $f = \sum_i f_i^* f_i$, $f \in \mathcal{A}$. Then $\deg f_i \leq N$, and there exist row vectors $a_i \in \mathbb{C}^{(N+1)(N+2)/2}$ such that $f_i = a_i w_N$. It implies that $f = \sum_i (a_i w_N)^* a_i w_N = w_N^* C w_N$, where $C = \sum_i a_i^* a_i$ is positive semidefinite.

On the other hand, if $f = w_N^* C w_N$, $C \geq 0$, then C is a sum of rank one positive semidefinite matrices. That is there exist row vectors $a_i \in \mathbb{C}^{(N+1)(N+2)/2}$ such that $C = \sum_i a_i^* a_i$, which implies that $f = \sum_i (a_i w_N)^* a_i w_N \in \sum \mathcal{A}^2$. \square

Proposition 4. Let $a, b \in \mathbb{R}$. The element $L := (x + x^*)^2 - 2a(x + x^*) + b$ is in $\sum \mathcal{A}^2$ if and only if $b \geq \frac{4a^2q}{(q+1)^2}$.

Proof. First suppose that $L \in \sum \mathcal{A}^2$. Then by Lemma 2 there is a positive semi-definite matrix $C = [c_{i,j}]_{i,j=\overline{1,3}}$ such that $L = w_1^* C w_1$. Comparing coefficients at $x^{*m} x^n$, $m+n \leq 2$ yields

$$\begin{aligned} b &= c_{11} \\ -2a &= c_{13} + c_{21} = c_{31} + c_{12} \\ 1 &= c_{23} = c_{32} \\ q+1 &= c_{22} + qc_{33} \end{aligned}$$

By multiplying these equations with $\alpha_1 = 1$, $\alpha_2 = \frac{2aq}{(1+q)^2}$, $\alpha_3 = 0$ and $\alpha_4 = \frac{4a^2q^2}{(1+q)^3}$, respectively, and adding them we derive

$$b - \frac{4a^2q}{(q+1)^2} = \text{tr} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_2 \\ \alpha_2 & \alpha_4 & \alpha_3 \\ \alpha_2 & \alpha_3 & q\alpha_4 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

Both matrices in the preceding equation are positive semidefinite. Therefore, since the trace of the product of two positive semidefinite matrices is nonnegative, $b - \frac{4a^2q}{(q+1)^2} \geq 0$.

Conversely, suppose that $b \geq 4a^2q(q+1)^{-2}$. Setting

$$f = q^{-1/2}(-2aq(1+q)^{-1} + qx + x^*),$$

we compute

$$(14) \quad (x + x^*)^2 - 2a(x + x^*) + \frac{4a^2q}{(q+1)^2} = f^* f.$$

Hence $L = f^* f + ((b - 4a^2q(q+1)^{-2})^{1/2} \mathbf{1})^2$. \square

The preceding result has the following interesting application.

Proposition 5. *Let $X \neq 0$ be a q -normal operator. Then each non-zero real number is a regular point for the symmetric operator $X + X^*$. In particular, $X + X^*$ is not essentially self-adjoint.*

Proof. Suppose that $a \in \mathbb{R} \setminus \{0\}$. It follows from (14) that the element

$$(x + x^*)^2 - 2a(x + x^*) + \frac{4a^2q}{(q+1)^2} = ((x + x^*) - a)^2 - a^2 \left(\frac{q-1}{q+1} \right)^2$$

is a square in A . This implies that

$$(15) \quad \|(X + X^* - a)\varphi\| \geq a \left| \frac{q-1}{q+1} \right| \|\varphi\|,$$

for $\varphi \in \mathcal{D}^\infty(X)$. Since $\mathcal{D}^\infty(X)$ is a core for $X + X^*$ by Lemma 1 (ii), the inequality (15) holds for all $\varphi \in \mathcal{D}(X + X^*)$. This shows that a is a regular point for $X + X^*$.

Let T denote the closure of $X + X^*$ and assume to the contrary that T is self-adjoint. The numbers of $\mathbb{R} \setminus \{0\}$ are regular points for $X + X^*$ and hence for the selfadjoint operator T . Therefore, $\mathbb{R} \setminus \{0\} \subseteq \rho(T)$, so that $\sigma(T) = \{0\}$. The latter implies that $X + X^* = 0$ which is impossible for $X \neq 0$. \square

For a positive Borel measure μ on \mathbb{R}_+ satisfying (4) we define a linear functional F_μ on \mathcal{A} by

$$(16) \quad F_\mu(f) = \langle \pi_\mu(f) \mathbf{1}_{\Delta_q}, \mathbf{1}_{\Delta_q} \rangle, \quad f \in \mathcal{A},$$

where π_μ is defined by (13).

Lemma 3. *Let π_μ be the $*$ -representation of \mathcal{A} defined by (13) and let $f = f^* \in \mathcal{A}$. Then we have $\pi_\mu(f) \geq 0$ if and only if*

$$(17) \quad F_\mu(g^*fg) \geq 0 \text{ for all } g \in \mathcal{A}.$$

Proof. Let C_μ be as in (6). It follows from relation (12) that the graph topology on $\mathcal{D}(\pi_\mu) = \mathcal{D}^\infty(C_\mu)$ is generated by the family of seminorms $\|C_\mu^n(\cdot)\|$, $n \in \mathbb{N}_0$. Set $\varphi_0 = \mathbf{1}_{\Delta_q}$. For $k \in \mathbb{Z}$ define

$$\varphi_k := \begin{cases} (X_\mu^k \varphi_0)(t), & \text{if } k \geq 0; \\ (X_\mu^{*|k|} \varphi_0)(t), & \text{if } k < 0. \end{cases}$$

Then $\varphi_k = q^{k/2} t^k \mathbf{1}_{\{q^{-k/2} \Delta_q\}}(t)$ if $k \geq 0$ and $\varphi_k = t^{|k|} \mathbf{1}_{\{q^{|k|/2} \Delta_q\}}(t)$ if $k < 0$. Since $X_\mu^* X_\mu = C_\mu^2$, for each $k \in \mathbb{Z}$ the set $\{p(X_\mu^* X_\mu) \varphi_k \mid p \in \mathbb{C}[t]\}$ is dense in the subspace $L_2(q^{-k/2} \Delta_q, d\mu) \subseteq \mathcal{D}(\pi_\mu)$ with respect to the graph topology.

Consider the case $\mu(\{0\}) = 0$. Then $\mathcal{D}(\pi_\mu)$ is a direct sum of $L_2(q^{k/2} \Delta_q, d\mu)$, $k \in \mathbb{Z}$. Hence φ_0 is cyclic for π_μ . Therefore, since $F_\mu(g^*fg) = \langle \pi_\mu(f) \pi_\mu(g) \varphi_0, \pi_\mu(g) \varphi_0 \rangle$ and $\pi_\mu(\mathcal{A}) \varphi_0$ is dense in $\mathcal{D}(\pi_\mu)$ in the graph topology, it follows that $\pi_\mu(f) \geq 0$ if and only if condition (17) is satisfied.

If $\text{supp } \mu = \{0\}$, then the statement is trivial. Consider the case $\text{supp } \mu \neq \{0\}$, $\mu(\{0\}) \neq 0$ and let μ_1 be the Borel measure on \mathbb{R}_+ defined by $\mu_1(\Delta) = \mu(\Delta \setminus \{0\})$. Then $\pi_\mu(x)$ is a direct

sum of 0 and $\pi_{\mu_1}(x)$. Let $f = f^* \in \mathcal{A}(q)$ satisfy (17). Since $F_\mu = F_{\mu_1}$, f is positive in π_1 . It suffices to prove $f(0, 0) \geq 0$. For let $f = \sum_{m,n} \alpha_{mn} x^{*m} x^n$ and $q > 1$. Then for $k \in \mathbb{N}$ we have

$$\begin{aligned} q^{-k} F_\mu(x^{*k} f x^k) &= q^{-k} \sum_{m,n} \alpha_{mn} \langle \pi_\mu(x^{*(k+m)} x^{k+n}) \varphi_0, \varphi_0 \rangle = \\ &= q^{-k} \sum_{m,n} \alpha_{mn} \langle X_\mu^{k+n} \mathbf{1}_{\Delta_q}, X_\mu^{k+m} \mathbf{1}_{\Delta_q} \rangle = \sum_n q^n \alpha_{nn} \int t^{2n} \mathbf{1}_{q^{-(k+n)/2} \Delta_q} dt \end{aligned}$$

which converges to $\alpha_{00} = f(0, 0)$, for $k \rightarrow \infty$. In the case $q < 1$ we have $q^k F_\mu(x^k f x^{*k}) \rightarrow f(0, 0)$, $k \rightarrow \infty$. It implies $f(0, 0) \geq 0$. \square

Remark. In the case $\mu(\{0\}) = 0$ we have seen in the preceding proof that the vector $\varphi_0 = \mathbf{1}_{\Delta_q}$ is cyclic for π_μ . Therefore, by uniqueness of GNS-representation (see Theorem 8.6.4. in [S4]), π_μ is unitarily equivalent to the GNS-representation of the positive functional F_μ .

Denote by $\mathcal{B} = \mathbb{C}[x^*x]$ the unital $*$ -subalgebra of \mathcal{A} generated by the single element x^*x . For every element $g \in \mathcal{B}$ there exists a unique polynomial, denoted by $g(t) \in \mathbb{C}[t]$, such that $g = g(x^*x)$. For $f = \sum_{m,n} \alpha_{mn} x^{*m} x^n \in \mathcal{A}(q)$ we define

$$\mathfrak{p}(f) := \sum_n \alpha_{nn} x^{*n} x^n = \sum_n \alpha_{nn} q^{n(n-1)/2} (x^*x)^n \in \mathcal{B}.$$

Then the mapping $\mathfrak{p} : \mathcal{A} \rightarrow \mathcal{B}$ is a conditional expectation as introduced in [SS]. We collect some properties of \mathfrak{p} in a lemma. We omit its simple proof.

Lemma 4. *Let $q > 0$.*

- (i) *For every positive functional F_μ defined by (16) and every $f \in \mathcal{A}$ holds $F_\mu(f) = F_\mu(\mathfrak{p}(f))$. In particular,*

$$F_\mu(f) = \int_{\Delta_q} (\mathfrak{p}(f))(t) d\mu(t^{1/2}).$$

- (ii) *For $f \in \mathcal{A}$ and $g_1, g_2 \in \mathcal{B}$, we have $\mathfrak{p}(g_1^* f g_2) = g_1^* \mathfrak{p}(f) g_2$.*

Proposition 6. *Let $\mathcal{H} = L_2(\mathbb{R}, d\lambda)$, where $d\lambda$ is the Lebesgue measure on \mathbb{R} . We define operators U_0 and C_0 on \mathcal{H} by*

$$(18) \quad (U_0 \varphi)(t) = \varphi(t+1), \quad (C_0 \psi)(t) = q^{t/2} \psi(t),$$

where $\mathcal{D}(U_0) = \mathcal{H}$, $\mathcal{D}(C_0) = \{\psi \mid t\psi(t) \in L_2(\mathbb{R}, d\lambda)\}$. Then $X_0 := U_0 C_0$ is a q -normal operator and an element $f = f^ \in \mathcal{A}$ is in \mathcal{A}_+ if and only if*

$$(19) \quad \langle f(X_0, X_0^*) \psi, \psi \rangle \geq 0 \text{ for all } \psi \in \mathcal{D}^\infty(X_0).$$

Proof. Let μ_0 be a measure on \mathbb{R}_+ satisfying (4) such that $\mu_0 \upharpoonright \Delta_q$ coincides with the Lebesgue measure $d\lambda$. Then the unitary operator $\mathcal{U} : L_2(\mathbb{R}, d\lambda) \rightarrow L_2(\Delta_q, d\mu_0)$, $(\mathcal{U}\psi)(t) = \psi(q^{t/2})$ defines a unitary equivalence of X_0 and X_{μ_0} . Hence X_0 is q -normal and for each $f \in \mathcal{A}_+$ condition (19) is satisfied.

Conversely, suppose that (19) holds. Let π_{μ_0} and F_{μ_0} be the $*$ -representation and positive functional defined by (13) and (16) respectively. Since (19) holds, we have $F_{\mu_0}(g^*fg) \geq 0$ for all $g \in \mathcal{A}$ and hence

$$F_{\mu_0}(h^*g^*fgh) \geq 0 \quad \text{for all } h \in \mathcal{B}, g \in \mathcal{A}(q).$$

Using Lemma 4 we obtain for a fixed $g \in \mathcal{A}$ and every $h \in \mathcal{B}$,

$$\begin{aligned} F_{\mu_0}(h^*g^*fgh) &= p_{\mu_0}(\mathfrak{p}(h^*g^*fgh)) = \int_{\Delta_q} h^*(t)(\mathfrak{p}(g^*fg))(t)h(t)d\lambda(t^{1/2}) = \\ &= \int_{\Delta_q} (\mathfrak{p}(g^*fg))(t)|h(t)|^2d\lambda(t^{1/2}) \geq 0. \end{aligned}$$

Since the polynomials $h \in \mathbb{C}[t]$ are dense in $L^2(\Delta_q, d\lambda)$ it follows that

$$(\mathfrak{p}(g^*fg))(t) \geq 0 \quad \text{for } t \in \Delta_q, g \in \mathcal{A}.$$

Let μ be another measure on \mathbb{R}_+ satisfying (4) and F_μ be the corresponding positive functional. We assume first that $\mu(\{0\}) = 0$. Then

$$F_\mu(g^*fg) = \int_{\Delta_q} (\mathfrak{p}(g^*fg))(t)d\lambda(t^{1/2}) \geq 0 \quad \text{for all } g \in \mathcal{A}$$

which implies that $\pi_\mu(f) \geq 0$. □

Theorem 2. For $c \in \mathbb{R}$, set $L_c := (x + x^*)^4 - 2(x + x^*)^2 + c$. Then:

(i) $L_c \in \sum \mathcal{A}^2$ if and only if $c \geq \frac{q(q+1)^2}{(q^2+1)^2}$,

(ii) there exists $\varepsilon > 0$ such that $L := (x + x^*)^4 - 2(x + x^*)^2 + \frac{q(q+1)^2}{(q^2+1)^2} - \varepsilon \in \mathcal{A}_+$.

Proof. (i): First suppose that $L_c \in \sum \mathcal{A}^2$. By Lemma 2 there exists a positive semidefinite 6×6 -matrix C such that $L = w_2^*Cw_2$. Comparing coefficients at $x^{*m}x^n$ in the equation $L = w_2^*Cw_2$, we obtain the following equations

$$\begin{aligned} c &= c_{1,1}, \\ 0 &= c_{1,2} + c_{3,1} = c_{2,1} + c_{1,3}, \\ -2 &= c_{1,4} + c_{3,2} + c_{6,1} = c_{4,1} + c_{2,3} + c_{1,6}, \\ -2q - 2 &= c_{1,5} + c_{2,2} + qc_{3,3} + c_{5,1}, \\ 0 &= c_{2,6} + c_{4,3} = c_{6,2} + c_{3,4}, \\ 0 &= qc_{3,5} + c_{2,4} + c_{5,2} + q^2c_{6,3} = qc_{5,3} + c_{4,2} + c_{2,5} + q^2c_{3,6}, \\ 1 &= c_{4,6} = c_{6,4}, \\ (1+q)(1+q^2) &= q^2c_{5,6} + c_{4,5} = q^2c_{6,5} + c_{5,4}, \\ (1+q^2)(1+q+q^2) &= c_{4,4} + qc_{5,5} + q^4c_{6,6}, \end{aligned}$$

Multiplying each line with the respective α_i ,

$$\begin{aligned}\alpha_1 &= 1, \\ \alpha_2 &= 0, \\ \alpha_3 &= 0, \\ \alpha_4 &= \frac{q+q^2}{(1+q^2)^2}, \\ \alpha_5 &= 0, \\ \alpha_6 &= 0, \\ \alpha_7 &= \frac{q^3(1+q)^2}{(-1+q)^2(1+q^2)^2(1+q+q^2)}, \\ \alpha_8 &= -\frac{q^2(1+q)^3}{(-1+q)^2(1+q^2)^3(1+q+q^2)}, \\ \alpha_9 &= \frac{q(1+q)^2}{(-1+q)^2(1+q^2)^2(1+q+q^2)},\end{aligned}$$

and adding them we get

$$c - \frac{q(1+q)^2}{(1+q^2)^2} = \text{Tr} \left(\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_3 \\ \alpha_2 & \alpha_4 & \alpha_3 & \alpha_6 & \alpha_6 & \alpha_5 \\ \alpha_2 & \alpha_3 & q\alpha_4 & \alpha_5 & q\alpha_6 & q^2\alpha_6 \\ \alpha_3 & \alpha_6 & \alpha_5 & \alpha_9 & \alpha_8 & \alpha_7 \\ \alpha_4 & \alpha_6 & q\alpha_6 & \alpha_8 & q\alpha_9 & q^2\alpha_8 \\ \alpha_3 & \alpha_5 & q^2\alpha_6 & \alpha_7 & q^2\alpha_8 & q^4\alpha_9 \end{bmatrix} C \right)$$

By some simple computations one checks that the matrix containing the α_i is positive semidefinite. Since C is also positive semidefinite, it follows from the preceding that

$$(20) \quad c \geq \frac{q(1+q)^2}{(1+q^2)^2}.$$

Conversely, suppose that (20) is satisfied. Setting

$$u_1(x, x^*) = -\frac{q}{1+q^2} + \frac{x^{*2}}{q(1+q)} + \frac{(1+q^2)x^*x}{1+q} + \frac{q^2x^2}{1+q}$$

and

$$u_2(x, x^*) = -\frac{q}{1+q^2} + \frac{x^{*2}}{1+q} + \frac{(1+q^2)x^*x}{1+q} + \frac{qx^2}{1+q},$$

we compute

$$(21) \quad (x+x^*)^4 - 2(x+x^*)^2 + \frac{q(1+q)^2}{(1+q^2)^2} = u_1^*u_1 + \frac{1+q+q^2}{q}u_2^*u_2.$$

This implies that $L \in \sum \mathcal{A}^2$ if (20) holds.

(ii): Since the element L remains invariant if we replace x by x^* and q by q^{-1} , it suffices to treat the case $q > 1$. Assume to the contrary that no such $\varepsilon > 0$ exists. Let X_0 be the q -normal operator from Proposition 6. Then there exists a sequence of unit vectors $\varphi_n \in \mathcal{D}^\infty(X_0)$ such that $\langle L(X_0, X_0^*)\varphi_n, \varphi_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. It follows from (21) that

$$(22) \quad u_1(X_0, X_0^*)\varphi_n \rightarrow 0 \text{ and } u_2(X_0, X_0^*)\varphi_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Put $X := X_0^2$. Then X is a q^4 -normal operator, $X^* = X_0^{*2}$ and $X_0^*X_0 = (qX_0^{*2}X_0^2)^{1/2} = q^{1/2}|X|$. Define operators

$$F_1 = \frac{q(q+1)}{(q-1)}(u_1(X_0, X_0^*) - u_2(X_0, X_0^*)) = X_0^{*2} - q^2X_0^2 = X^* - q^2X,$$

$$F_2 = \frac{1}{q-1}(qu_1(X_0, X_0^*) - u_2(X_0, X_0^*)) = qX + \frac{1+q^2}{1+q}q^{1/2}|X| - \frac{q}{1+q^2}.$$

Then $F_1\varphi_n \rightarrow 0$ and $F_2\varphi_n \rightarrow 0$. Let $X = UC$ be the polar decomposition of X . From Proposition 1 we get $X^* - q^2X = q^2U^*(I - U^2)C$ which implies that

$$(23) \quad (I - U^2)C\varphi_n \rightarrow 0.$$

Put $\alpha = q$, $\beta = q^{1/2}\frac{1+q^2}{1+q}$, $\gamma = -\frac{q}{1+q^2}$, so that $F_2 = \alpha UC + \beta C + \gamma$. Then $(I - U^2)F_2\varphi_n \rightarrow 0$. Combined with (23) it follows that

$$(24) \quad (I - U^2)\varphi_n \rightarrow 0.$$

Let $\mathbb{T}_+ = \{e^{it}, t \in [-\pi/2, \pi/2)\}$, $\mathbb{T}_- = \{e^{it}, t \in [\pi/2, 3\pi/2)\}$. We put $\xi_n = E_U(\mathbb{T}_+)\varphi_n$, $\psi_n = E_U(\mathbb{T}_-)\varphi_n$. Then $\varphi_n = \psi_n + \xi_n$ and (24) yields

$$(25) \quad (U - I)\xi_n \rightarrow 0 \quad \text{and} \quad (U + I)\psi_n \rightarrow 0.$$

Since C is positive, $C + 1$ is invertible, so that $(C + 1)^{-1}F_2\varphi_n \rightarrow 0$. By $UC = q^2CU$,

$$(C + 1)^{-1}(\alpha q^2CU + \beta C + \gamma)\varphi_n \rightarrow 0.$$

Using (25) we obtain

$$\frac{\alpha q^2 C}{C + 1}(\xi_n - \psi_n) + \frac{\beta C + \gamma}{C + 1}(\xi_n + \psi_n) \rightarrow 0$$

which implies that

$$\frac{(\alpha q^2 + \beta)C + \gamma}{C + 1}\xi_n - \frac{(\alpha q^2 - \beta)C - \gamma}{C + 1}\psi_n \rightarrow 0.$$

Since $q > 1$, $\alpha q^2 - \beta > 0$ and $\gamma < 0$. Hence the operator $(\alpha q^2 - \beta)C - \gamma$ has a bounded inverse. Applying this inverse to the preceding equation we get

$$(26) \quad -\psi_n + \frac{(\alpha q^2 + \beta)C + \gamma}{(\alpha q^2 - \beta)C - \gamma}\xi_n \rightarrow 0.$$

Applying U and using Proposition 1 and (25) we derive

$$(27) \quad \psi_n + \frac{(\alpha q^2 + \beta)q^2C + \gamma}{(\alpha q^2 - \beta)q^2C - \gamma}\xi_n \rightarrow 0.$$

Adding (26) and (27) we obtain

$$(28) \quad \frac{\alpha_1 C^2 + \beta_1 C + \gamma_1}{((\alpha q^2 - \beta)q^2C - \gamma)((\alpha q^2 - \beta)C - \gamma)}\xi_n \rightarrow 0, \quad n \rightarrow \infty,$$

where $\alpha_1 = 2q^2(\alpha^2 q^4 - \beta^2)$, $\beta_1 = -2\beta\gamma(1 + q^2)$, $\gamma_1 = -2\gamma^2$. The polynomial $\alpha_1 c^2 + \beta_1 c + \gamma_1$ has two real roots $c_1 < 0 < c_2$.

Since $X = (U_0 C_0)^2 = q^{-1/2} U_0^2 C_0^2$, we get $U = U_0^2$, $C = q^{-1/2} C_0^2$. By (18),

$$(U\varphi)(t) = \varphi(t+2), \quad (C\psi)(t) = q^{t-1/2}\psi(t) \quad \text{for } \varphi \in L_2(\mathbb{R}, d\lambda), \psi \in \mathcal{D}(C_0^2).$$

Set $t_2 = \log_q c_2 + 1/2$. Then (28) implies that

$$\|\xi_n\|^2 - \int_{t_2-1/2}^{t_2+1/2} |\xi_n|^2 dt \rightarrow 0.$$

Hence $\langle U\xi_n, \xi_n \rangle = \int_{\mathbb{R}} \xi_n(t+2) \overline{\xi_n(t)} dt \rightarrow 0$. Combined with (25) this yields $\xi_n \rightarrow 0$. Further, (26) implies $\psi_n \rightarrow 0$ which contradicts $\|\xi_n + \psi_n\| = \|\varphi_n\| = 1$. \square

We end up the section with the following

Proposition 7. *Let $f \in \mathbb{R}[t]$. If $f(x^*x) \in \mathcal{A}(q)_+$, then $f(x^*x) \in \sum \mathcal{A}(q)^2$.*

Proof. Let us choose a measure μ satisfying (4) such that $\text{supp } \mu = \mathbb{R}_+$ and let X_μ be the operator defined by (5). Then the spectrum of $X_\mu^* X_\mu$ is equal to \mathbb{R}_+ . Therefore, since $f(X_\mu^* X_\mu) \geq 0$, we have $f \geq 0$ on \mathbb{R}_+ . Hence (see e.g. [M]) there exist polynomials $g_1, g_2 \in \mathbb{C}[t]$ such that $f(t) = g_1(t)^* g_1(t) + t \cdot g_2(t)^* g_2(t)$. Then

$$\begin{aligned} f(x^*x) &= g_1(x^*x)^* g_1(x^*x) + x^*x \cdot g_2(x^*x)^* g_2(x^*x) = \\ &= g_1(x^*x)^* g_1(x^*x) + x^* \cdot g_2(qx^*x)^* g_2(qx^*x) \cdot x \in \sum \mathcal{A}(q)^2. \end{aligned}$$

\square

4. THE COMPLEX q -MOMENTS PROBLEM AND FORMALLY q -NORMAL OPERATORS

Definition 5. A linear functional F on \mathcal{A} is called a q -moment functional if there exists a well-behaved $*$ -representation π of \mathcal{A} and a vector $\xi \in \mathcal{D}(\pi)$ such that

$$(29) \quad F(a) = \langle \pi(a)\xi, \xi \rangle \quad \text{for all } a \in \mathcal{A}.$$

Then the q -moment problem asks:

When is a given functional F on \mathcal{A} a q -moment functional?

In this formulation the q -moment problem is a *generalized moment problem* in the sense of [S5]. Next we give two reformulations of the q -moment problem.

Since $\{x^{*k}x^l; k, l \in \mathbb{N}_0\}$ is a vector space basis of \mathcal{A} , there is a one-to-one-correspondence between complex 2-sequences and linear functionals on \mathcal{A} given by $F_a(x^{*k}x^l) = a_{kl}$, $k, l \in \mathbb{N}_0$, where $a = (a_{kl})_{k,l \in \mathbb{N}_0}$ is a 2-sequence. The definition of a well-behaved representation (Definition 4) yields the following equivalent formulation of the q -moment problem:

Given a 2-sequence $(a_{kl})_{k,l \in \mathbb{N}_0}$, does there exist a q -normal operator X and a vector $\xi \in \mathcal{D}^\infty(X)$ such that

$$(30) \quad a_{kl} = \langle X^{*k}X^l\xi, \xi \rangle \quad \text{for all } k, l \in \mathbb{N}_0?$$

Before we turn to the second reformulation we consider an example.

Example. Suppose that μ is a positive Borel measure on Δ_q . Denote by $\tilde{\mu}$ the unique extension of μ to a measure on \mathbb{R}_+ satisfying (4). Let X_μ be the q -normal operator defined by (5) and $\xi \in \mathcal{D}^\infty(X_\mu)$. Then there is a q -moment functional defined by $F_{\mu,\xi}(f(x, x^*)) := \langle f(X_\mu, X_\mu^*)\xi, \xi \rangle$. Since

$$(X_\mu^{*k} X_\mu^l \xi)(t) = q^{(l^2+l-k^2+k-2kl)/4} t^{k+l} \xi(q^{(l-k)/2} t), \quad k, l \in \mathbb{N}_0,$$

by Lemma 1, the corresponding q -moments are

$$\begin{aligned} a_{kl} &= F_{\mu,\xi}(x^{*k} x^l) = \langle X_\mu^{*k} X_\mu^l \xi, \xi \rangle = \int_{\mathbb{R}_+} q^{(l^2+l-k^2+k-2kl)/4} t^{k+l} \xi(q^{(l-k)/2} t) \overline{\xi(t)} d\mu(t) \\ (31) \quad &= q^{(l^2+l-k^2+k-2kl)/4} \int_{\mathbb{R}_+} (q^{k/2} t)^{k+l} \xi(q^{l/2} t) \overline{\xi(q^{k/2} t)} d\tilde{\mu}(q^{k/2} t) = \\ &= q^{(l^2+l+k^2+k)/4} \int_{\mathbb{R}_+} t^{k+l} \xi(q^{l/2} t) \overline{\xi(q^{k/2} t)} d\tilde{\mu}(t). \end{aligned}$$

Using Theorem 1 and formula (31) we obtain another equivalent formulation of the q -moment problem in terms of measures and integrals:

Given a 2-sequence $(a_{kl})_{k,l \in \mathbb{N}_0}$, does there exist a family $\mu_i, i \in I$, of positive Borel measures on Δ_q and a vector $\xi = (\xi_i) \in \bigoplus_i \mathcal{D}^\infty(X_{\mu_i})$ in the Hilbert space $\bigoplus_i L^2(\mathbb{R}_+, \tilde{\mu}_i)$ such that

$$a_{kl} = \sum_i q^{(l^2+l+k^2+k)/4} \int_{\mathbb{R}_+} t^{k+l} \xi_i(q^{l/2} t) \overline{\xi_i(q^{k/2} t)} d\tilde{\mu}_i(t) \text{ for } k, l \in \mathbb{N}_0?$$

The next theorem is the counter-part of Haviland's theorem from the classical moment problem. For this we need the following

Definition 6. A linear functional F on \mathcal{A} is said to be *positive* if $F(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and *it strongly positive* if $F(a) \geq 0$ for all $a \in \mathcal{A}_+$.

Each strongly positive functional is positive, but Proposition 4 below shows that the converse is not true.

Theorem 3. *A linear functional F on \mathcal{A} is a q -moment functional if and only if F is strongly positive.*

Proof. From the definition of the cone \mathcal{A}_+ (Definition 3) it is obvious that q -moment functionals are strongly positive.

Suppose that F is strongly positive. To prove that F is a q -moment functional we need some preparations. First we define some auxiliary algebras.

Let \mathcal{F} be the $*$ -algebra of all Borel functions $f(t)$ on \mathbb{R}_+ which are polynomially bounded (that is, there exists a polynomial $p \in \mathbb{C}[t]$ such that $|f(t)| \leq p(t)$ for $t \in \mathbb{R}_+$). We denote by \mathfrak{K} the $*$ -algebra generated by an element u and the $*$ -algebra \mathcal{F} with defining relations

$$(32) \quad u^*u = uu^* = 1, \quad uf(t) = f(q^{1/2}t), \quad f(t)u^* = f(q^{1/2}t),$$

for $f \in \mathcal{F}$. Clearly, \mathfrak{X} has a vector space basis $\{x^n c^k; k \in \mathbb{N}_0, n \in \mathbb{Z}\}$, where $c^2 = x^*x$ and $x^{-n} := x^{*n}$ for $n < 0$, $n \in \mathbb{Z}$. Hence there is an injective $*$ -homomorphism J of \mathcal{A} into \mathfrak{X} given by $J(x) = uf_0$, where $f_0(t) = t$. We identify $J(a)$ and a for $a \in \mathcal{A}$ and consider \mathcal{A} as a $*$ -subalgebra of \mathfrak{X} . With a slight abuse of notation we shall write $x = ut$, where t means the function $f_0(t) = t$ on \mathbb{R}_+ .

Let μ be a Borel measure on \mathbb{R}_+ satisfying (4). Then

$$\begin{aligned} (\pi_\mu(f)\varphi)(t) &= f(q^{1/2}t)\varphi(q^{1/2}t), \quad (u\varphi)(t) = \varphi(q^{1/2}t), \\ \varphi \in \mathcal{D}(\pi_\mu) &= \{\varphi \in L^2(\mathbb{R}_+, \mu) \mid t^n \varphi \in L^2(\mathbb{R}_+, \mu) \text{ for all } n \in \mathbb{N}\}, \end{aligned}$$

defines a $*$ -representation of \mathfrak{X} on $\mathcal{H} = L^2(\mathbb{R}_+, d\mu)$ and $\overline{\pi_\mu(ut)} = X_\mu$ is the q -normal operator given by (5). Setting

$$\mathfrak{X}_+ := \{x \in \mathfrak{X} \mid \pi_\mu(x) \geq 0 \text{ for all measures } \mu \text{ satisfying (4)}\},$$

we clearly have $\mathcal{A}_+ = \mathfrak{X}_+ \cap \mathcal{A}$.

Let \mathfrak{X}_b be the $*$ -subalgebra of \mathfrak{X} generated by u and the subset \mathcal{F}_b of all $f \in \mathcal{F}$ of compact support and consider the $*$ -subalgebra $\mathcal{Y} = \mathcal{A} + \mathfrak{X}_b$ of \mathfrak{X} . Clearly, \mathcal{A}_+ is cofinal in $\mathcal{Y}_+ := \mathcal{Y} \cap \mathcal{X}$, that is, for each $y \in \mathcal{Y}_+$ there exists $a \in \mathcal{A}_+$ such that $a - y \in \mathcal{Y}_+$. Therefore, since $\mathcal{A}_+ = \mathfrak{X}_+ \cap \mathcal{A}$, F extends to a linear functional, denoted again by F , such that $F(y) \geq 0$ for all $y \in \mathcal{Y}_+$. Let π_F denote the $*$ -representation of \mathcal{Y} with cyclic vector φ obtained by the GNS construction from the functional F (see e.g. [S4], Section 8.6). Then, by the GNS-construction,

$$(33) \quad F(y) = \langle \pi_F(y)\varphi, \varphi \rangle \text{ for } y \in \mathcal{Y}.$$

Let $\mathbf{1}_\Delta$ be the characteristic function of a Borel subset $\Delta \subseteq \mathbb{R}_+$ and define $(E(\Delta)f)(t) = \overline{\pi_F(\mathbf{1}_\Delta)f(t)}$, $f \in \mathcal{H}(\pi_F)$. Then E defines a spectral measure on \mathbb{R}_+ . Let $U = \overline{\pi_F(u)}$. From (32) it follows that $UE(\Delta)U^* = E(q^{-1/2}\Delta)$. Let $C = \int_0^\infty \lambda dE(\lambda)$. Then $X := UC$ is a q -normal operator on $\mathcal{H}(\pi_F)$ by Proposition 1. The proof is complete once we have shown that $\pi_F(x) \subseteq X$, or equivalently,

$$(34) \quad \pi_F(a)\varphi \in \mathcal{D}(X) \text{ and } \pi_F(x)\pi_F(a)\varphi = X\pi_F(a)\varphi \text{ for } a \in \mathcal{A}.$$

Indeed, because X is q -normal, by Lemma 1 and Definition 4 there is a well-behaved $*$ -representation π of \mathcal{A} on $\mathcal{D}(\pi) := \cap_n \mathcal{D}(X^n)$ such that $\pi(x) = X[\mathcal{D}(\pi)]$. The relation $\pi_F(x) \subseteq X$ implies that $\pi_F \subseteq \pi$. Therefore, by (33), $F(a) = \langle \pi_F(a)\varphi, \varphi \rangle = \langle \pi(a)\varphi, \varphi \rangle$ for $a \in \mathcal{A}$, so F is a q -moment functional.

Let $f \in \mathcal{F}_b$ and $k \in \mathbb{Z}$. Then the operator $\pi_F(f)$ is bounded and we have $f(C) = \int_0^\infty f(\lambda)dE(\lambda) = \overline{\pi_F(f)}$ by the spectral calculus. Therefore, since $u^k f(t) = f(q^{k/2}t)u^k$ by (32), we have

$$\begin{aligned} C\pi_F(u^k f(t)) &= C\pi_F(f(g^{k/2}t))\pi_F(u^k) = Cf(g^{k/2}t)\pi_F(u^k) \\ (35) \quad &= \pi_F(tf(g^{k/2}t))\pi_F(u^k) = \pi_F(tu^k f(t)). \end{aligned}$$

We prove (34) for $a = u^{\tau n}t^n$, where $\tau = \pm 1$ and $n \in \mathbb{N}_0$. Let $\varepsilon > 0$ be fixed. We choose $\alpha_\varepsilon > 0$ such that $t^{2n} \leq \varepsilon(1 + t^{2n+2})$ for $t > \alpha_\varepsilon$ and denote the characteristic function of the

interval $[0, \alpha_\varepsilon]$ by $\mathbf{1}_\varepsilon$. Setting $g_\varepsilon(t) = \mathbf{1}_\varepsilon(t)t^n$ and $f_\varepsilon(t) = t^n - g_\varepsilon(t)$, we have $g_\varepsilon \in \mathcal{F}_b$ and $f_\varepsilon(t)^2 \leq \varepsilon(1 + t^{2n+2})$ for $t \in \mathbb{R}_+$. Hence

$$(36) \quad \varepsilon(t^2 + t^{2n+4}) - t^2 f_\varepsilon(t)^2 \in \mathcal{Y}_+, \quad \varepsilon(1 + t^{2n+2}) - f_\varepsilon(t)^2 \in \mathcal{Y}_+.$$

Now we compute

$$(37) \quad \begin{aligned} & \|\pi_F(x)\pi_F(u^{\tau n}t^n)\varphi - X\pi_F(u^{\tau n}g_\varepsilon)\varphi\|^2 = \|\pi_F(ut)\pi_F(u^{\tau n}t^n)\varphi - \pi_F(u)C\pi_F(u^{\tau n}g_\varepsilon)\varphi\|^2 = \\ & = \|\pi_F(u)(\pi_F(tu^{\tau n}t^n)\varphi - \pi_F(tu^{\tau n}g_\varepsilon)\varphi)\|^2 = \|\pi_F(q^{-n/2}u^{\tau(n+1)})t(t^n - g_\varepsilon(t))\varphi\|^2 = \\ & = q^{-n} \|\pi_F(tf_\varepsilon(t))\varphi\|^2 = q^{-n} \langle \pi_F(t^2 f_\varepsilon(t)^2)\varphi, \varphi \rangle = q^{-n} F(t^2 f_\varepsilon(t)^2) \leq \varepsilon q^{-n} F(t^2 + t^{2n+4}). \end{aligned}$$

Here we used first equations (35) and (32), then the fact that $\pi_F(u^{\tau(n+1)})$ preserves the norm and equation (33) for $y = t^2 f_\varepsilon(t)^2$. Since F is \mathcal{Y}_+ -positive, we have $F(\varepsilon(t^2 + t^{2n+4}) - t^2 f_\varepsilon(t)^2) \geq 0$ by (36) which gives the inequality in the last line.

Using now the fact that $\varepsilon(1 + t^{2n+2}) - f_\varepsilon(t)^2 \in \mathcal{Y}_+$ by (35) we derive

$$(38) \quad \begin{aligned} & \|\pi_F(u^{\tau n}t^n)\varphi - \pi_F(u^{\tau n}g_\varepsilon)\varphi\|^2 = \|\pi_F(u^{\tau n}f_\varepsilon)\varphi\|^2 \\ & = \|\pi_F(f_\varepsilon(t)\varphi)\|^2 = F(f_\varepsilon(t)^2) \leq \varepsilon F(1 + t^{2n+2}). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, (4) and (4) imply that

$$\pi_F(u^{\tau n}g_\varepsilon)\varphi \rightarrow \pi_F(u^{\tau n}t^n)\varphi \quad \text{and} \quad X\pi_F(u^{\tau n}g_\varepsilon)\varphi \rightarrow \pi_F(x)\pi_F(u^{\tau n}t^n)\varphi.$$

Therefore, since X is closed, we have $\pi_F(x)\pi_F(u^{\tau n}t^n)\varphi \in \mathcal{D}(X)$ and $\pi_F(x)\pi_F(u^{\tau n}t^n)\varphi = X\pi_F(u^{\tau n}t^n)\varphi$. This proves (34) for $a = u^{\tau n}t^n$. Since these elements span \mathcal{A} , (34) holds for all $a \in \mathcal{A}$ which completes the proof. \square

Theorem 4. *There exists a positive linear functional on \mathcal{A} which is not a q -moment functional.*

Before we prove this we state two technical lemmas. The first one is taken from [S3], Lemma 2.

Lemma 5. *Let \mathcal{A} be a unital $*$ -algebra which has a faithful $*$ -representation π and is the union of a sequence of finite dimensional subspaces E_n , $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists a number $k_n \in \mathbb{N}$ such that the following is satisfied: if $a \in \sum \mathcal{A}^2$ is in E_n , then we can write a as a finite sum $\sum_j a_j^* a_j$ such that all a_j are in E_{k_n} .*

Then the cone $\sum \mathcal{A}^2$ is closed in \mathcal{A} with respect to the finest locally convex topology on \mathcal{A} .

Lemma 6. *Suppose that π is a well-behaved representation such that $\pi(x) \neq 0$. Then π is faithful.*

Proof. Since π is well-behaved, there is a q -normal operator X such that $\pi(x) = X \upharpoonright \mathcal{D}^\infty(X)$. By Theorem 1, X is a direct sum of operators X_{μ_i} . Since $\pi(x) \neq 0$, $X_{\mu_i} \neq 0$ for one i .

Suppose that $f(x, x^*) \in \mathcal{A}$, $f \neq 0$. It suffices to prove that there exists a vector $\varphi \in \mathcal{D}^\infty(X_{\mu_i})$ such that $f(X_{\mu_i}, X_{\mu_i}^*)\varphi \neq 0$.

The polar decomposition $X_{\mu_i} = U_{\mu_i} C_{\mu_i}$ is given by (6). From Proposition 1(iv) it follows that there are polynomials $f_k \in \mathbb{C}[t]$, $k = -n, \dots, n$ such that $f(X_{\mu_i}, X_{\mu_i}^*) = \sum_{k=-n}^n U_{\mu_i}^k f_k(C_{\mu_i})$. Since $f \neq 0$, there is a j such that $f_j \neq 0$. Put $\varphi = \mathbf{1}_{q^{m/2}\Delta_q}(t)$ and choose $m \in \mathbb{Z}$ such that the

interval $q^{m/2}\Delta_q$ contains no zero of $f_j(t)$. Then, since $\mu_i \neq 0$, we have $\mu_i(q^{m/2}\Delta_q) \neq 0$ by (4) and hence $\varphi \neq 0$. Using (6) we calculate

$$f(X_\mu^*, X_\mu)\varphi = \sum_{k=-n}^n U_\mu^k f_k(t) \mathbb{1}_{q^{m/2}\Delta_q}(t) = \sum_{k=-n}^n f_k(q^{k/2}t) \mathbb{1}_{q^{(m-k)/2}\Delta_q}(t).$$

If the latter would be zero, then $f_k(q^{k/2}t) \mathbb{1}_{q^{(m-k)/2}\Delta_q}(t) \equiv 0$ in $L^2(\mathbb{R}_+, d\mu)$ for all k , in particular for $k = j$ which is a contradiction. Thus $f(X_\mu^*, X_\mu)\varphi \neq 0$. \square

Proof of Theorem 4: We denote by E_k the subspace of elements $f \in \mathcal{A}$, $\deg f \leq k$. Obviously, $\sum g_i^* g_i \in E_{2k}$ implies that $g_i \in E_k$ for all i . By Lemma 6 \mathcal{A} has a faithful representation. Therefore, Lemma 5 applies, so the cone $\sum \mathcal{A}(q)^2$ is closed in the finest locally convex topology.

By Theorem 2 there exists an element $L \in \mathcal{A}_+$ such that $L \notin \sum \mathcal{A}^2$. Since $\sum \mathcal{A}^2$ is closed, by the separation theorem for convex sets there is a linear functional F on \mathcal{A} such that $F(L) < 0$ and $F(\sum \mathcal{A}(q)^2) \geq 0$. By the latter condition, F is a positive linear functional. Since F is not strongly positive (by $F(L) < 0$), it is not a moment functional by Theorem 3. \square

Definition 7. A densely defined operator X on a Hilbert space H is a *formally q -normal operator* if $\mathcal{D}(X) \subseteq \mathcal{D}(X^*)$ and $\|Xf\| = \sqrt{q}\|X^*f\|$ for $f \in \mathcal{D}(X)$.

It is well-known [C] that there exist formally normal operators which have no normal extensions in larger Hilbert spaces. The next theorem shows that a similar result holds for formally q -normal operators.

Theorem 5. *There exists a formally q -normal operator X which has no q -normal extension in a possibly larger Hilbert space.*

Proof. We retain the notation from the proof of Theorem 4. Let π_F denote the GNS representation of F with cyclic vector φ , see [S4]. Then $F(a) = \langle \pi_F(a)\varphi, \varphi \rangle$ for $a \in \mathcal{A}$.

We show that $X := \pi_F(x)$ is a formally q -normal operator which has no q -normal extension. Indeed, since π_F is a $*$ -representation of \mathcal{A} , we have $\mathcal{D}(X) = \mathcal{D}(\pi_F) = \mathcal{D}(\pi_F(x^*)) \subseteq \mathcal{D}(\pi_F(x)^*) = \mathcal{D}(X^*)$. For $\psi \in \mathcal{D}(X)$ we have

$$\begin{aligned} \|X\psi\|^2 &= \langle \pi_F(x)\psi, \pi_F(x)\psi \rangle = \langle \pi_F(x^*x)\psi, \psi \rangle = q^{-1} \langle \pi_F(xx^*)\psi, \psi \rangle = \\ &= \langle X^*\psi, X^*\psi \rangle = q^{-1} \|X^*\psi\|^2. \end{aligned}$$

Assume that Y is a q -normal operator on a (possibly larger) Hilbert space such that $X \subseteq Y$. Then $L(X, X^*) \subseteq L(Y, Y^*)$ and hence

$$\langle L(Y, Y^*)\varphi, \varphi \rangle = \langle L(X, X^*)\varphi, \varphi \rangle = \langle \pi_F(L)\varphi, \varphi \rangle = F(L) < 0.$$

Since $L \in \mathcal{A}_+$, this is a contradiction. \square

5. A STRICT POSITIVSTELLENSATZ FOR q -POLYNOMIALS

The strict Positivstellensatz (Theorem 6) proved in this section can be viewed as a q -analogue of the Reznick's Positivstellensatz [R].

Let $f = \sum_{i,j} a_{ij} x^{*i} x^j \in \mathcal{A}$ and $\deg f = m$. We denote by $f_m = \sum_{\{i+j=m\}} a_{ij} x^{*i} x^j$ the highest order degree part of f . We write f_m as

$$f_m = \sum_{r=0}^{\lfloor m/2 \rfloor} b_r (x^* x)^r x^{m-2r} + \sum_{r=0}^{\lfloor m/2 \rfloor - 1} b_{-r} x^{*(m-2r)} (x^* x)^r, \quad b_r \in \mathbb{C}.$$

The symbol of f is the function $\sigma_f(\omega, \bar{\omega})$ on $\mathbb{C} \setminus \{0\}$ defined by

$$(39) \quad \sigma_f(\omega, \bar{\omega}) := \sum_{r=0}^{\lfloor m/2 \rfloor} b_r |\omega|^{-r(m-2r)} \omega^{\frac{m}{2}-r} + \sum_{r=0}^{\lfloor m/2 \rfloor - 1} b_{-r} \bar{\omega}^{\frac{m}{2}-r} |\omega|^{-r(m-2r)}.$$

Let \mathcal{N} denote the set consisting of $\mathbf{1}$ and all finite products of elements $q^k x^* x + 1$, where $k \in \mathbb{Z}$.

Theorem 6. *Let $f = f^* \in \mathcal{A}$, $\deg f = 4m$, $m \in \mathbb{N}$. Suppose that:*

(i) *For every q -normal operator X there exists a $\varepsilon_X > 0$ such that*

$$\langle f(X, X^*) \varphi, \varphi \rangle \geq \varepsilon_X \langle \varphi, \varphi \rangle, \quad \varphi \in \mathcal{D}^\infty(X),$$

(ii) *$\sigma_f(\omega, \bar{\omega}) > 0$ for all $\omega \in \mathbb{T} := \{z \in \mathbb{C} : |z| = q^{1/2}\}$.*

Then there exists an element $b \in \mathcal{N}$ such that $b^ f b \in \sum \mathcal{A}(q)^2$.*

The proof of this theorem follows a similar pattern as the proof of the strict Positivstellensatz for the Weyl algebra given in [S1]. We first recall a basic definition and a result from [S1], see e.g. [S2].

A unital $*$ -algebra \mathcal{Y} is called *algebraically bounded* if for each element $y \in \mathcal{B}$ there exists a $\lambda_y > 0$ such that

$$(40) \quad \lambda_y \cdot \mathbf{1} - y^* y \in \sum \mathcal{Y}^2.$$

Lemma 7. *Let \mathcal{Y} be an algebraically bounded $*$ -algebra and $y = y^* \in \mathcal{B}$. If*

$$(41) \quad \langle \pi(y) \varphi, \varphi \rangle > 0 \quad \text{for all } \varphi \in \mathcal{H}_\pi, \quad \varphi \neq 0,$$

for $$ -representation π of \mathcal{Y} , then $y \in \sum \mathcal{Y}^2$.*

Proof. Assume to the contrary that $y \notin \sum \mathcal{X}^2$. Since \mathcal{Y} is algebraically bounded, $\mathbf{1}$ is an internal point of the wedge $\sum \mathcal{Y}^2$. Therefore, by the Eidelheit separation theorem for convex sets [K], there exists a linear functional F on \mathcal{Y} such that $F(y) \leq 0$, $F(\mathbf{1}) > 0$, and $F(\sum \mathcal{Y}^2) \geq 0$. If π_F denotes the GNS-representation of F with cyclic vector φ , then $F(y) = \langle \pi_F(y) \varphi, \varphi \rangle \leq 0$. Since $F(\mathbf{1}) = \|\varphi\|^2 > 0$, the latter contradicts (41). \square

The proof of the theorem will be divided into three steps.

I. Let ρ be a fixed well-behaved representation of \mathcal{A} such that $\rho(x) \neq 0$. By Lemma 6, ρ is faithful. For notational simplicity we identify $a \in \mathcal{A}$ with $\rho(a)$. Then \bar{x} is a q -normal operator and \mathcal{A} becomes a $*$ -algebra of operators acting on the invariant dense domain $\mathcal{D}(\rho) = \mathcal{D}^\infty(\bar{x})$. Define the following operators

$$y_k := x^2 (q^k x^* \bar{x} + 1)^{-1}, \quad v_k := x (q^k x^* \bar{x} + 1)^{-1}, \quad z_k := (q^k x^* \bar{x} + 1)^{-1}, \quad k \in \mathbb{Z}.$$

Is easily checked that y_k, y_k^*, v_k, v_k^* are bounded operators which map the domain $\mathcal{D}^\infty(\bar{x})$ into itself. Let \mathfrak{X} be the $*$ -algebra of operators on $\mathcal{D}^\infty(\bar{x})$ generated by $x, x^*, y_k, y_k^*, v_k, v_k^*, z_k, k \in \mathbb{Z}$. Then the following relations hold in \mathfrak{X} :

$$(42) \quad xz_k = v_k, \quad z_k x^* = v_k^*, \quad x^2 z_k = y_k, \quad z_k x^{*2} = y_k^*$$

$$(43) \quad xz_k = z_{k+1}x, \quad x^* z_k = z_{k-1}x^*$$

$$(44) \quad z_k z_m = z_m z_k, \quad z_k^* = z_k,$$

$$(45) \quad q^{2k+1} y_k^* y_k + q^k v_k^* v_k + z_k = 1, \quad q^k v_k^* v_k + z_k^2 = z_k,$$

$$(46) \quad y_m^* y_k = y_k^* y_m, \quad y_k^* y_k = q^{-4} y_{k-2} y_{k-2}^*, \quad v_k v_k^* = q v_{k+1}^* v_{k+1},$$

$$(47) \quad y_k z_m = z_{m+2} y_k, \quad y_k^* z_m = z_{m-2} y_k^*,$$

$$(48) \quad v_k z_m = z_{m+1} v_k, \quad v_k^* z_m = z_{m-1} v_k^*,$$

$$(49) \quad y_m y_k^* = q y_{k+1}^* y_{m+1}, \quad v_k z_m = v_m z_k,$$

$$(50) \quad y_k y_m = y_{m+2} y_{k-2}, \quad y_m^* y_k^* = y_{k-2}^* y_{m+2}^*,$$

$$(51) \quad q^k z_k (1 - z_m) = q^m z_m (1 - z_k),$$

$$(52) \quad q^{k+m+1} y_k^* y_m = (1 - z_k)(1 - z_m), \quad k, m \in \mathbb{Z}.$$

Let \mathfrak{Y} denote the subalgebra of \mathfrak{X} generated by $\mathbf{1}, y_k, y_k^*, v_k, v_k^*, z_k$, where $k \in \mathbb{Z}$. From (45) it follows that condition (40) holds for the algebra generators $y = y_k, y_k^*, v_k, v_k^*, z_k$ of \mathfrak{Y} . Hence \mathfrak{Y} is an algebraically bounded $*$ -algebra by Lemma 2.1 in [S1].

Our next aim is to study representations of \mathfrak{Y} .

II. Suppose that π is a non-zero $*$ -representation of \mathfrak{Y} on a Hilbert space \mathcal{H}_π . Since \mathfrak{Y} is algebraically bounded, all operators $\pi(y)$, $y \in \mathfrak{Y}$, are bounded, so we can assume that $\mathcal{D}(\pi) = \mathcal{H}_\pi$. Let $\mathcal{H}_0 = \pi(z_0)$, $\mathcal{H}_1 = \ker \pi(\mathbf{1} - z_0)$. By (51) we have $\ker \pi(z_k) = \mathcal{H}_0$ and $\ker \pi(\mathbf{1} - z_k) = \mathcal{H}_1$, $k \in \mathbb{Z}$. From (45) it follows that $\pi(v_k) \upharpoonright \mathcal{H}_0 = 0$. The third relation in (46) yields $\pi(v_k^*) \upharpoonright \mathcal{H}_0 = 0$. Further, (48) implies that \mathcal{H}_0 is invariant under $\pi(y_k), \pi(y_k^*)$, $k \in \mathbb{Z}$. It follows from (45) that $\pi(v_k), \pi(v_k^*), \pi(y_k), \pi(y_k^*)$ restricted onto \mathcal{H}_1 are 0. The preceding implies that \mathcal{H}_0 and \mathcal{H}_1 are invariant subspaces of the representation π . Let $\pi = \pi_0 \oplus \pi_1 \oplus \pi_2$ be the corresponding decomposition of π on $\mathcal{H}_\pi = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

Now we analyze the three subrepresentations π_0, π_1 and π_2 . We begin with π_0 . By construction of π_0 we have $\pi_0(z_k) = \pi_0(v_k) = \pi_0(v_k^*) = 0$ for $k \in \mathbb{Z}$. From (45) we obtain $\pi_0(y_k^* y_k) = 1/q^{2k+1}$. By (46), $\pi_0(y_k y_k^*) = q^4 \pi_0(y_{k+2}^* y_{k+2}) = 1/q^{2k+1}$, so that $\pi_0(q^{k+1/2} y_k)$ is unitary. Finally, it follows from (52) that all operators $\pi_0(q^{k+1/2} y_k)$ coincide. Hence there exists a unitary operator Y on \mathcal{H}_0 such that

$$(53) \quad \pi_0(y_k) = q^{-k-\frac{1}{2}} Y, \quad k \in \mathbb{Z}.$$

Next we consider π_1 . As noted above, $\pi_1(\mathbf{1}) = \pi_1(z_k) = I_{\mathcal{H}_1}$ and $\pi_1(v_k) = \pi_1(v_k^*) = \pi_1(y_k) = \pi_1(y_k^*) = 0$.

Finally, we turn to π_2 . It is convenient to introduce the notation

$$Z_k := \pi_2(z_k), \quad V_k := \pi_2(v_k), \quad Y_k := \pi_2(y_k).$$

Then $\ker Z_k = \ker(I - Z_k) = \{0\}$ by the construction of π_2 . Combined with (45) we conclude that $0 < Z_k < I$. Further, (52) implies that all operators $q^{-k}(Z_k^{-1} - 1)$, $k \in \mathbb{Z}$, are equal and positive. Set $C := q^{-k/2}(Z_k^{-1} - 1)^{1/2}$. Then $Z_k = q^k C^2 + 1$ and using (45) we get $|V_k| = C(q^k C^2 + 1)^{-1}$. Let $V_k = U_k |V_k|$ be the polar decomposition of V_k . Note that $\ker C = \{0\}$. From (49) we derive

$$U_k C(q^k C^2 + 1)^{-1} (q^m C^2 + 1)^{-1} = U_m C(q^m C^2 + 1)^{-1} (q^k C^2 + 1)^{-1}, \quad k, m \in \mathbb{Z}.$$

This implies that all operators U_k , $k \in \mathbb{Z}$, are equal. Set $U := U_k$. Since $\ker V_k = \ker V_k^* = \{0\}$, U is unitary. By (48) we have

$$UC(q^k C^2 + 1)^{-1} (q^m C^2 + 1)^{-1} = (q^{m+1} C^2 + 1)^{-1} UC(q^k C^2 + 1)^{-1}, \quad k, m \in \mathbb{Z},$$

and since $C(q^k C^2 + 1)^{-1}$ is invertible,

$$U(q^m C^2 + 1)^{-1} = (q^{m+1} C^2 + 1)^{-1} U.$$

The latter is equivalent to $UC^2 U^* = qC^2$. Therefore, by Proposition 1, $X := UC = V_k Z_k^{-1}$ is a q -normal operator. Further, π_2 leaves $\mathcal{D}^\infty(X) = \mathcal{D}^\infty(C)$ invariant. Indeed, by construction this is true for the generators and hence for all elements of \mathcal{Y} .

Let $h \in \mathfrak{X}$. Using the relations (43) and $(q^k x^* x + 1)z_k = \mathbf{1}_{\mathfrak{X}}$ it follows that h is of the form $h = h_1 z_{k_1} z_{k_2} \dots z_{k_m}$, where $h_1 \in \mathcal{A}$ and $k_1, \dots, k_m \in \mathbb{Z}$. Since the operators z_{k_1}, \dots, z_{k_m} map $\mathcal{D}^\infty(\overline{x})$ bijectively onto itself, $h = 0$ if and only if $h_1 = 0$. Therefore, the $*$ -representation π_2 gives rise to a unique $*$ -representation $\tilde{\pi}_2$ of \mathcal{X} on $\mathcal{D}^\infty(X)$ defined by $\tilde{\pi}_2(y) = \pi_2(y) \upharpoonright \mathcal{D}^\infty(X)$ for $y \in \mathcal{Y}$,

$$\tilde{\pi}_2(x) = X \upharpoonright \mathcal{D}^\infty(X), \quad \text{and} \quad \tilde{\pi}_2(z_k) = (q^k X^* X + 1)^{-1} \upharpoonright \mathcal{D}^\infty(X), \quad k \in \mathbb{Z}.$$

III. Now let f be as in Theorem 6 and let $f_{4m} = \sum_{i+j=4m} a_{ij} x^{*i} x^j$ be its highest degree part. From (42) and (43) it follows that

$$y := z_0^m f(x, x^*) z_0^m \in \mathfrak{Y}.$$

Our next aim is to apply Lemma 7 in order to conclude that $y \in \sum \mathcal{Y}^2$.

Let $\pi = \pi_0 \oplus \pi_1 \oplus \pi_2$ be a representation of \mathfrak{Y} as analyzed above.

First we determine $\pi_0(y)$. Suppose $i, j \in \mathbb{N}_0$ and $i + j < 4m$. Applying the relations (42) and (43) it follows that $z_0^m x^{*i} x^j z_0^m = w_1 w_2 \dots w_s$, where each w_l is equal to one of the elements $y_k, y_k^*, v_k, v_k^*, z_0$. Since $i + j = 4m$, not all elements w_l can be equal to some y_k . Therefore, since $\pi_0(z_k) = \pi_0(v_k) = \pi_0(v_k^*) = 0$, we obtain $\pi_0(z_0^m x^{*i} x^j z_0^m) = 0$. Hence $\pi_0(y) = \pi_0(z_0^m f z_0^m) = \pi_0(z_0^m f_{4m} z_0^m)$. Now we write

$$(54) \quad f_{4m} = \sum_{k=0}^{2m} b_k (x^* x)^{2m-k} x^{2k} + \sum_{k=1}^{2m} b_{-k} x^{*2k} (x^* x)^{2m-k}.$$

For the monomial $(x^*x)^{2m-k}x^{2k}$ we treat the cases $k \leq m$ and $k > m$. First suppose that $k \leq m$. Using relations (42)-(44) we compute

$$\begin{aligned} z_0^m (x^*x)^{2m-k} x^{2k} z_0^m &= z_0^m (x^*x)^{2m-k} z_{2k}^{m-k} x^{2k} z_0^k = \\ &= (z_0 x^* x)^m (x^* x z_{2k})^{m-k} x^{2k} z_0^k = \\ &= (\mathbf{1} - z_0)^m (q^{-2k} (\mathbf{1} - z_{2k}))^{m-k} (x^2 z_{2k-2}) (x^2 z_{2k-4}) \dots (x^2 z_0) = \\ &= (\mathbf{1} - z_0)^m (q^{-2k} (\mathbf{1} - z_{2k}))^{m-k} y_{2k-2} y_{2k-4} \dots y_0. \end{aligned}$$

Applying π_0 to both sides and using (53) we derive

$$\begin{aligned} \pi_0(z_0^m (x^*x)^{2m-k} x^{2k} z_0^m) &= q^{-2k(m-k)} \pi_0(y_{2k-2} y_{2k-4} \dots y_0) = \\ &= q^{-2k(m-k)} q^{-k^2+k/2} Y^k = |q^{1/2}|^{-2k(2m-k)} (q^{1/2} Y)^k. \end{aligned}$$

In the case $k > m$ we obtain

$$\begin{aligned} z_0^m (x^*x)^{2m-k} x^{2k} z_0^m &= z_0^{2m-k} (x^*x)^{2m-k} z_0^{k-m} x^{2k} z_0^m = \\ &= (z_0 x^* x)^{2m-k} (z_0^{k-m} x^{2(k-m)}) (x^{2m} z_0^m) = \\ &= (\mathbf{1} - z_0)^{2m-k} (x^2 z_{-2}) (x^2 z_{-4}) \dots (x^2 z_{-2(k-m)}) \times \\ &\quad \times (x^2 z_{2(m-1)}) (x^2 z_{2(m-2)}) \dots (x^2 z_0) = \\ &= (\mathbf{1} - z_0)^{2m-k} y_{-2} y_{-4} \dots y_{-2(k-m)} \cdot y_{2(m-1)} y_{2(m-2)} \dots y_0. \end{aligned}$$

Again, applying π_0 and using (53) we get

$$\begin{aligned} \pi_0(z_0^m (x^*x)^{2m-k} x^{2k} z_0^m) &= \\ &= \pi_0(y_{-2} y_{-4} \dots y_{-2(k-m)}) \pi_0(y_{2(m-1)} y_{2(m-2)} \dots y_0) = \\ &= q^{(k-m)(k-m+1) - \frac{k-m}{2}} Y^{k-m} q^{-m(m-1) - \frac{m}{2}} Y^m = |q^{1/2}|^{-2k(2m-k)} (q^{1/2} Y)^k. \end{aligned}$$

Proceeding in a similar manner we derive

$$\pi_0(z_0^m x^{*2k} (x^*x)^{2m-k} z_0^m) = |q^{1/2}|^{-2k(2m-k)} (q^{1/2} Y^*)^k.$$

Let $Y = \int_{\mathbb{T}} \omega dE(\omega)$ be the spectral decomposition of the unitary operator Y . Comparing the preceding computations with the definition of σ_f we get $\pi_0(y) = \int_{\mathbb{T}} \sigma_f(q^{1/2}\omega, q^{1/2}\bar{\omega}) dE(\omega)$. From assumption (ii) it follows that there exists $\varepsilon > 0$ such that $\sigma_f(q^{1/2}\omega, q^{1/2}\bar{\omega}) \geq \varepsilon$ for $\omega \in \mathbb{T}$. Hence

$$\langle \pi_0(y)\psi, \psi \rangle = \int_{\mathbb{T}} \sigma_f(q^{1/2}\omega, q^{1/2}\bar{\omega}) d\langle E(\omega)\psi, \psi \rangle \geq \varepsilon \|\psi\|^2, \quad \psi \in \mathcal{H}_0.$$

Applying assumption (i) to the q -normal operator $X = 0$ yields $a_{00} = f(0, 0) > 0$. By (42) and (43) we have $\pi_1(y) = \pi(a_{00} z_0^{2m}) = a_{00} \cdot I$.

Finally we turn to π_2 . As shown above, there exists a $*$ -representation $\tilde{\pi}_2$ of the $*$ -algebra \mathfrak{K} on $\mathcal{D}^\infty(X)$ such that $\tilde{\pi}_2 \upharpoonright \mathcal{A}$ is well-behaved and $\tilde{\pi}_2(y) = \pi_2(y) \upharpoonright \mathcal{D}^\infty(X)$ for $y \in \mathcal{Y}$. Using

assumption (i) of the theorem we obtain for $\zeta \in \mathcal{D}^\infty(X)$,

$$\begin{aligned} \langle \pi_2(y)\zeta, \zeta \rangle &= \langle \pi_2(z_0^m f z_0^m)\zeta, \zeta \rangle = \langle \tilde{\pi}_2(z_0^m f z_0^m)\zeta, \zeta \rangle = \\ &= \langle \tilde{\pi}_2(f)\tilde{\pi}_2(z_0^m)\zeta, \tilde{\pi}_2(z_0^m)\zeta \rangle \geq \varepsilon_X \|\tilde{\pi}_2(z_0^m)\zeta\|^2 = \varepsilon_X \|\pi_2(z_0^m)\zeta\|^2. \end{aligned}$$

Since $\pi_2(y)$ and $\pi_2(z_0)$ are bounded operators and $\ker \pi_2(z_0) = \{0\}$ by construction, we conclude that $\langle \pi_2(y)\zeta, \zeta \rangle > 0$ for all $\zeta \in \mathcal{H}_2$, $\zeta \neq 0$.

Since $\pi = \pi_0 \oplus \pi_1 \oplus \pi_2$, it follows from the preceding analysis that $\langle \pi(y)\psi, \psi \rangle > 0$ for all $\psi \in \mathcal{H}_\pi$, $\psi \neq 0$. Therefore, by Lemma 7,

$$g = \sum_{i=1}^r g_i^* g_i \in \sum \mathfrak{Y}^2.$$

The relations (43) imply that in the algebra \mathcal{X} each $g_i \in \mathcal{Y}$ can be written $g_i = f_i h_i$, where $f_i \in \mathcal{A}$ and h_i is a finite product of elements z_j . That is, $h_i^{-1} \in \mathcal{N} \subseteq \mathcal{A}$. Multiplying both sides of the equation

$$g = z_0^m f z_0^m = \sum_{i=1}^r (f_i h_i)^* f_i h_i$$

by $(h_1 h_2 \dots h_r z_0^m)^{-1}$ from the left and from the right we obtain

$$bfb = \sum_{i=1}^r \tilde{f}_i^* \tilde{f}_i \in \mathcal{A}(q)^2,$$

where $\tilde{f}_i = f_i h_i (h_1 h_2 \dots h_r z_0^m)^{-1} \in \mathcal{A}$ and $b = (h_1 h_2 \dots h_r)^{-1} \in \mathcal{N}$. □

The next example illustrates the assertion of the strict Positivstellensatz. Its proof is analogous to the proof of Theorem 2.

Proposition 8. *Suppose that $q = 1/2$. Then:*

- (i) $L := (xx^*)^2 - (x + x^*)^2 + 3.7 \notin \sum \mathcal{A}^2$,
- (iii) L satisfies the assumptions (i) and (ii) in Theorem 6,
- (ii) $(1 + qx^*)L(1 + qx^*) \in \sum \mathcal{A}^2$.

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